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The continuous spectrum of the Orr–Sommerfeld equation. Part 2. Eigenfunction expansions

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The expansion of an arbitrary two-dimensional solution of the linearized stream-function equation in terms of the discrete and continuum eigenfunctions of the Orr–Sommerfeld equation is discussed for flows in the half-space, $y \in [0, \infty)$. A recent result of Salwen is used to derive a biorthogonality relation between the solution of the linearized equation for the stream function and the solutions of the adjoint problem.

For the case of temporal stability, the orthogonality relation obtained is equivalent to that of Schensted for bounded flows. This relationship is used to carry out the formal solution of the initial-value problem for temporal stability. It is found that the vorticity of the disturbance at $t = 0$ is the proper initial condition for the temporal stability problem. Finally, it is shown that the set consisting of the discrete eigenmodes and continuum eigenfunctions is complete.

For the spatial stability problem, it is shown that the continuous spectrum of the Orr–Sommerfeld equation contains four branches. The biorthogonality relation is used to derive the formal solution to the boundary-value problem of spatial stability. It is shown that the boundary-value problem of spatial stability requires the stream function and its first three partial derivatives with respect to x to be specified at $x = 0$ for all t. To be applicable to practical problems, this solution will require modification to eliminate disturbances originating at $x = \infty$ and travelling *upstream* to $x = 0$.

For the special case of a constant base flow, the method is used to calculate the evolution in time of a particular initial disturbance.

1. Introduction

Recent calculations of the discrete eigenmodes of the Orr–Sommerfeld equation (Jordinson 1971; Mack 1976; Corner, Houston & Ross 1976; Murdock & Stewartson 1977) have indicated that, for a given Reynolds number and wavenumber (frequency), the Orr–Sommerfeld equation for Blasius flow has only a finite number of discrete temporal (spatial) eigenfunctions. Since a finite set of functions cannot be complete, these calculations raised the question of how to expand the stream function of an arbitrary disturbance in terms of the normal modes. These authors suggested that, in addition to the finite discrete spectrum which they found, there is a continuous spectrum.

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In part 1 (Grosch & Salwen 1978*a*), we dealt with the existence of the continuous spectrum and the form of the related eigenfunctions for both the temporal and spatial problems. We showed that the Orr–Sommerfeld equation, for any mean shear flow approaching a constant velocity in the far field, possesess a continuous spectrum; we gave formulae for the location of the temporal and spatial continua in the complex wave-speed plane; and we calculated the temporal continuum eigenfunctions for some particular cases. In this paper, we turn our attention to the use of the discrete and continuum eigenfunctions of the Orr–Sommerfeld equation to calculate the temporal or spatial evolution of an arbitrary solution of the linear disturbance equations.

In a recent critique of the application of stability theory to the prediction of transition, Berger & Aroesty (1977) point out that, on the basis of the limited experimental evidence that is available, the coupling of free-stream disturbances to disturbances in the boundary layer appears to be extraordinarily weak and extremely selective in frequency and wavenumber. Mack (1977) makes the same point in a different way. He points out that, 'if there were no disturbances [inside the boundary layer], there would be no transition and the boundary layer would remain laminar. Consequently, it is futile to talk about transition without in some way bringing in the disturbances which cause it...'. Mack adds, '...the precise mechanism by which, say, free stream turbulence, sound, and different types of roughness cause transition remains to be discovered'.

The most detailed discussion of this problem appears to be that of Obremski, Morkovin & Landahl (1969). They consider various possible mechanisms by which sound or vorticity waves in the free stream might interact with the boundary layer and cause transition. On the basis of the available experimental evidence, they conclude that only a small portion of the external disturbance field excites Tollmien–Schlichting (TS) waves in the boundary layer and a significant portion appears to travel within the boundary layer with little or no interaction. The (unstated) conclusion seems to be that the mechanism which couples free-stream disturbances to a boundary layer and, thereby, initiates transition is unknown.

The central problem here is the solution of the general initial and boundary-value problems for disturbances to boundary-layer flow: how, given the form of the disturbance at a time $t = 0$, to find its variation with time and how, given the form of the disturbance at all times on a plane, $x = 0$, perpendicular to the boundary layer, to find out the way in which it propagates downstream. In this paper, we approach these problems, in the approximation obtained by assuming parallel flow and linearizing with respect to the disturbances, by expressing the solution as a sum over the discrete normal modes plus an integral over the continuum eigenfunctions of the Orr–Sommerfeld equation. If the (discrete plus continuum) eigenfunctions form a complete set, this approach will yield a valid solution of the problem.

Starting with Haupt (1912), a number of authors have dealt with the completeness of the set of temporal eigenfunctions in a bounded domain. Haupt showed that the eigenfunctions for two-dimensional disturbances to plane Couette flow form a complete set and Schensted (1960) proved completeness for the eigenfunctions for two-dimensional disturbances to plane Poiseuille flow and for axisymmetric disturbances to Poiseuille flow in a circular pipe. Yudovich (1965) and DiPrima & Habetler (1969) have proved the completeness of the eigenmodes for a large class of bounded flows.

We are unaware of any work on the completeness of the spatial eigenfunctions or, previous to this paper, on the completeness of the temporal eigenfunctions in an unbounded domain.

In § 2 we formulate the stability problem for two-dimensional disturbances to a parallel shear flow, $U(y)$, $0 \leqslant y < \infty$, in terms of the linearized equation for the stream function and boundary conditions. We next formulate the adjoint problem. A new result of Salwen (1979) is then used to derive a pseudocontinuity relation involving solutions of the linearized equation for the stream function and the adjoint solutions. This relation is then used to find the general biorthogonality condition for wave-like disturbances to the flow. The biorthogonality relation is specialized to the cases of temporal and spatial stability. The orthogonality relation for the temporal stability problem is that derived by Schensted (1960) and discussed by Reid (1965).

The temporal stability problem is considered in detail in § 3. The solution is Fourier analysed with respect to x. Then the formal solution of the initial-value problem for the temporal stability of a two-dimensional disturbance to a parallel shear flow is expressed as an expansion in terms of the eigenfunctions. The expansion coefficients are determined by inner products between the initial disturbance and the eigenfunctions of the adjoint equation. We show that the disturbance vorticity at $t = 0$ is the proper initial condition for the temporal stability problem.

In § 4 we examine the question of the completeness of the set of expansion functions for the temporal stability problem. Very recently, Gustavsson (1979) has treated the temporal initial-value problem by using Fourier–Laplace transforms. He finds poles in the transform plane which correspond to the discrete TS modes and a branch cut which corresponds to the continuous spectrum. We show in this section that the Fourier–Laplace transform solution of Gustavsson is identical to our Fourier transform, eigenfunction expansion solution for the initial-value problem of temporal stability. We therefore conclude that our expansion set is complete.

The spatial stability problem is considered in detail in § 5. The solution is Fourier analysed in t. The formulae for the four branches of the continuous spectrum of the spatial stability problem are derived and discussed. The formal solution of the boundary-value problem for the spatial stability of a two-dimensional disturbance to a parallel shear flow is expressed as an expansion in terms of the spatial eigenfunctions. The expansion coefficients are determined by inner products between the boundary conditions at $x = 0$ and the eigenfunctions of the adjoint equation. The boundary conditions at $x = 0$ are discussed. We have not yet been able to prove completeness for the set of expansion functions of the spatial stability problem.

In § 6, we apply the results of § 3 to the simple case of a constant base flow. In this case, we find the eigenfunctions and calculate and discuss the temporal evolution of a particular initial disturbance.

2. The linearized, two-dimensional Navier–Stokes equations: the bi-orthogonality relation

2.1. *Formulation of the problem*

The basic flow under consideration is a parallel shear flow, $U(y)$, in the semi-infinite region, $y \geqslant 0$. We are concerned with the temporal or spatial development of an 'infinitesimal', two-dimensional disturbance to this flow, $(u(x,y,t), v(x,y,t), 0)$. In this case, u and v can be expressed in terms of a stream function, $\Psi(x,y,t)$, by

$$u = \partial\Psi/\partial y, \quad v = -\partial\Psi/\partial x, \tag{1), (2}$$

and the linearized Navier–Stokes equations reduce to a single partial differential equation,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\Psi - \frac{d^2U}{dy^2}\frac{\partial\Psi}{\partial x} - \frac{1}{R}\nabla^4\Psi = \mathscr{L}\Psi = 0, \tag{3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{4}$$

In addition, Ψ must satisfy two boundary conditions at $y = 0$,

$$\left.\frac{\partial\Psi}{\partial x}\right|_{x,0,t} = -v(x,0,t) = 0 \tag{5}$$

and

$$\left.\frac{\partial\Psi}{\partial y}\right|_{x,0,t} = u(x,0,t) = 0, \tag{6}$$

and a 'finiteness' condition

$$\int_0^\infty \left[\left|\frac{\partial\Psi}{\partial x}\right|^2 + \left|\frac{\partial\Psi}{\partial y}\right|^2\right] dy = \int_0^\infty [|u|^2 + |v|^2]\,dy < \infty. \tag{7}$$

As a consequence of (7), Ψ must satisfy boundary conditions at infinity,

$$\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y} \to 0 \quad \text{as} \quad y \to +\infty. \tag{8}$$

For fixed x and t, $\Psi(x,y,t)$ belongs to a manifold, M, of functions, $\phi(y)$, satisfying

$$\phi, \; \frac{d\phi}{\partial y}, \; \frac{d^2\phi}{dy^2}, \; \frac{d^3\phi}{dy^3}, \; \frac{d^4\phi}{dy^4} \quad \text{all defined on } [0,\infty), \tag{9}$$

$$\phi, \; \frac{d\phi}{dy}, \; \frac{d^2\phi}{dy^2}, \; \frac{d^3\phi}{dy^3}, \quad \text{continuous on } [0,\infty), \tag{10}$$

$$\phi(0) = 0, \quad \phi'(0) = 0 \tag{11}$$

and

$$\int_0^\infty |\phi(y)|^2\,dy, \quad \int_0^\infty \left|\frac{d\phi}{dy}\right|^2 dy \quad \text{both exist.} \tag{12}$$

The continuum eigenfunctions which will be discussed in §§3 and 5 do not satisfy (12). Instead they belong to a manifold $M' \supset M$ of functions satisfying (9)–(11) and a weakened condition,

$$\phi(y) \quad \text{and} \quad \frac{d\phi}{dy} \quad \text{bounded in } [0,\infty). \tag{13}$$

We define an inner product,

$$(f,g) \equiv \int_0^\infty f^*(y)\,g(y)\,dy, \tag{14}$$

in M. The asterisk denotes the complex conjugate. This inner product is defined for the full Hilbert space of functions satisfying (12) and, in that space, has the usual properties of inner products.

2.2. *The adjoint problem*

For functions $f, g \in M$ we define the adjoint, $\mathscr{L}^+$, of $\mathscr{L}$ in the usual way by

$$\iiint \{f(x,y,t)\}^* \{\mathscr{L}^+ g(x,y,t)\}\, dx\, dy\, dt$$
$$= \iiint \{\mathscr{L} f(x,y,t)\}^* \{g(x,y,t)\}\, dx\, dy\, dt + \text{boundary terms}. \tag{15}$$

The definition of the adjoint used here yields an adjoint operator which is identical to the formal adjoint (Friedman 1969, pp. 2, 3).

An adjoint stream function, $\tilde{\Psi}(x,y,t)$, is a solution of the adjoint equation (with $U^* = U$),

$$\mathscr{L}^+\tilde{\Psi} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\tilde{\Psi} + 2\frac{dU}{dy}\frac{\partial^2\tilde{\Psi}}{\partial x\,\partial y} + \frac{1}{R}\nabla^4\tilde{\Psi} = 0, \tag{16}$$

with the boundary conditions at $y = 0$,

$$\left.\frac{\partial\tilde{\Psi}}{\partial x}\right|_{x,0,t} = \left.\frac{\partial\tilde{\Psi}}{\partial y}\right|_{x,0,t} = 0, \tag{17}$$

and the finiteness condition

$$\int_0^\infty \left\{\left|\frac{\partial\tilde{\Psi}}{\partial x}\right|^2 + \left|\frac{\partial\tilde{\Psi}}{\partial y}\right|^2\right\} dy = \int_0^\infty \{|\tilde{u}|^2 + |\tilde{v}|^2\}\, dy < \infty. \tag{18}$$

As above, equation (19) implies that $\tilde{\Psi}$ must satisfy boundary conditions

$$\frac{\partial\tilde{\Psi}}{\partial x}, \frac{\partial\tilde{\Psi}}{\partial y} \to 0 \quad \text{as} \quad y \to \infty. \tag{19}$$

When, as below, we look for solutions to the linearized stream-function equation (3) which have a wave-like behaviour in x and t, equation (3) reduces to the Orr–Sommerfeld equation and equation (16) reduces to the adjoint Orr–Sommerfeld equation. Our adjoint Orr–Sommerfeld equation is the complex conjugate of the adjoint equation derived by Schensted (1960) and quoted by Reid (1965). The reason for this difference is that we define the inner product in the usual way, (14), while Schensted's definition of the inner product (f, g) involves f instead of f^*.

2.3. *Bi-orthogonality*

Salwen (1979) has shown that the solutions of the linearized, three-dimensional Navier–Stokes equations, $\mathbf{u}$, p and the adjoint solutions $\tilde{\mathbf{u}}$, $\tilde{p}$ satisfy a 'continuity' equation

$$\frac{\partial\hat{\rho}}{\partial t} + \nabla\cdot\mathbf{J} = 0, \tag{20}$$

where

$$\hat{\rho} = \tilde{\mathbf{u}}^*\cdot\mathbf{u}, \tag{21}$$

$$\mathbf{J} = (\tilde{\mathbf{u}}^*\cdot\mathbf{u})\,\mathbf{U} + \frac{1}{R}\sum_{i=1}^{3}[(\nabla\tilde{u}_i^*)\,u_i - \tilde{u}_i^*(\nabla u_i)] + \tilde{\mathbf{u}}^* p + \mathbf{u}\tilde{p}^*, \tag{22}$$

and, as before, the asterisk denotes a complex conjugate.

For the two-dimensional disturbances considered here we will introduce two new inner products. Let Ψ be any solution of the original problem and $\tilde{\Psi}$ be any solution of the adjoint problem, then define

$$\langle \tilde{\Psi}, \Psi \rangle \equiv \int_0^\infty \hat{p}\, dy = \int_0^\infty \left(\frac{\partial \tilde{\Psi}^*}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial \tilde{\Psi}^*}{\partial y} \frac{\partial \Psi}{\partial y} \right) dy, \tag{23}$$

and

$$[\![\tilde{\Psi}, \Psi]\!] \equiv \int_0^\infty J_x\, dy, \tag{24}$$

with J_x the x component of $\mathbf{J}$. Using (22) and expressing $\mathbf{u}$, p and $\tilde{\mathbf{u}}$, $\tilde{p}$ in terms of Ψ and $\tilde{\Psi}$, it can be shown that

$$\begin{aligned} [\![\tilde{\Psi}, \Psi]\!] = \int_0^\infty \Bigg\{ & \frac{1}{R} \left[\tilde{\Psi}^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial \tilde{\Psi}^*}{\partial x} \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \tilde{\Psi}^*}{\partial x^2} \frac{\partial \Psi}{\partial x} - \frac{\partial^3 \tilde{\Psi}^*}{\partial x^3} \Psi + 2 \frac{\partial^2 \tilde{\Psi}^*}{\partial x\, \partial y} \frac{\partial \Psi}{\partial y} - 2 \frac{\partial \tilde{\Psi}^*}{\partial y} \frac{\partial^2 \Psi}{\partial x\, \partial y} \right] \\ & - \left[\tilde{\Psi}^* \frac{\partial^2 \Psi}{\partial t\, \partial x} + \frac{\partial^2 \tilde{\Psi}^*}{\partial t\, \partial x} \Psi \right] \\ & - U \left[\tilde{\Psi}^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \tilde{\Psi}^*}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial^2 \tilde{\Psi}^*}{\partial x^2} \Psi - 2 \frac{\partial \tilde{\Psi}^*}{\partial y} \frac{\partial \Psi}{\partial y} - \frac{\partial^2 \tilde{\Psi}^*}{\partial y^2} \Psi \right] \Bigg\} dy. \end{aligned} \tag{25}$$

The form of these inner products has been determined by the equations for the stream function and the adjoint stream function. However, we can use equations (23) and (25) to calculate inner products $\langle f, g \rangle$ and $[\![f, g]\!]$, evaluated at fixed x and t, of any functions $f(x, y, t)$ and $g(x, y, t)$.

It is straightforward to show that $\langle f, g \rangle$ is defined for the full Hilbert space of functions which satisfy equation (7) and, in that space, has the usual properties of inner products. On the other hand, $[\![f, f]\!]$ is not positive definite. This is due to the fact that it is possible to have wavelike solutions to equation (3) which propagate in either the upstream $(-x)$ or downstream $(+x)$ direction.

With these definitions it is easy to show that

$$\frac{\partial}{\partial t} \langle \tilde{\Psi}, \Psi \rangle + \frac{\partial}{\partial x} [\![\tilde{\Psi}, \Psi]\!] = 0, \tag{26}$$

for any solutions of the original and adjoint problems.

If Ψ and $\tilde{\Psi}$ are wave disturbances of the form

$$\Psi_{\alpha'\omega'} = \phi_{\alpha'\omega'}(y)\, e^{i(\alpha' x - \omega' t)}, \tag{27}$$

$$\tilde{\Psi}_{\alpha\omega} = \tilde{\phi}_{\alpha\omega}(y)\, e^{i(\alpha x - \omega t)}, \tag{28}$$

equation (26) reduces to

$$(\omega' - \omega^*) \langle \tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha'\omega'} \rangle = (\alpha' - \alpha^*) [\![\tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha'\omega'}]\!]. \tag{29}$$

This equation may be used to derive bi-orthogonality relations for the eigenfunctions of both the temporal and spatial stability problems.

For the temporal stability problem, α is real and given and α' equals α. The orthogonality relation for the temporal stability problem is then

$$(\omega' - \omega^*) \langle \tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha\omega'} \rangle = 0, \tag{30}$$

so the solutions of the temporal stability problem and the adjoint solutions are orthogonal unless $\omega' = \omega^*$. The orthogonality condition, equation (30), can be recognized as being essentially equivalent to that derived by Schensted (1960, p. 27, equation (2.2.3)), and discussed by Reid (1965). The only difference is that Schensted's adjoint solution is the complex conjugate of ours.

In the case of spatial stability, ω is real and given and $\omega' = \omega$ and the orthogonality relation is

$$(\alpha' - \alpha^*) [\![\tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha'\omega}]\!] = 0. \tag{31}$$

Thus, unless $\alpha' = \alpha^*$ the spatial eigenfunctions and adjoint eigenfunctions are orthogonal with the inner product defined by equation (25).

3. The temporal stability problem

3.1. *The eigenvalues and eigenfunctions*

For the temporal stability problem we modify the finiteness condition, equation (7), to

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left\{\left|\frac{\partial\Psi}{\partial x}\right|^2+\left|\frac{\partial\Psi}{\partial y}\right|^2\right\}dx\,dy < \infty. \tag{7'}$$

This ensures that the Fourier integral expansion of Ψ,

$$\Psi(x,y,t) = \int_{-\infty}^{\infty} \psi_\alpha(y,t)\, e^{i\alpha x}\, d\alpha, \tag{32}$$

exists. If we assume that ψ_α is of the form

$$\psi_\alpha(y,t) = \phi_\alpha(y)\, e^{-i\omega t}, \tag{33}$$

then ϕ_α is a solution of the Orr–Sommerfeld equation

$$\left\{L_\alpha^2 - i\alpha R\left[(U-c)\,L_\alpha - \frac{d^2U}{dy^2}\right]\right\}\phi_\alpha = 0, \tag{34}$$

with

$$c = \omega/\alpha, \quad L_\alpha \equiv \frac{d^2}{dy^2} - \alpha^2. \tag{35), (36}$$

Similarly we assume that the adjoint solution, $\tilde{\Psi}$, also satisfies equation (7′), thus ensuring that the Fourier integral expansion of $\tilde{\Psi}$,

$$\tilde{\Psi}(x,y,t) = \int_{-\infty}^{\infty} \tilde{\psi}_\alpha(y,t)\, e^{i\alpha x}\, d\alpha, \tag{37}$$

exists. It is assumed that $\tilde{\psi}_\alpha$ is of the form

$$\tilde{\psi}_\alpha(y,t) = \tilde{\phi}_\alpha(y)\, e^{-i\omega^* t}, \tag{38}$$

with $\tilde{\phi}_\alpha$ the solution of the adjoint Orr–Sommerfeld equation

$$\left\{L_\alpha^2 + i\alpha R\left[(U-c^*)\,L_\alpha + 2\frac{dU}{dy}\frac{d}{dy}\right]\right\}\tilde{\phi}_\alpha = 0, \tag{39}$$

with

$$c^* = \omega^*/\alpha. \tag{40}$$

Both ϕ_α and $\tilde{\phi}_\alpha$ satisfy the boundary conditions

$$\phi_\alpha(0) = \phi_\alpha'(0) = \tilde{\phi}_\alpha(0) = \tilde{\phi}_\alpha'(0) = 0, \tag{41}$$

and either

$$\phi_\alpha \to \phi_\alpha' \to \tilde{\phi}_\alpha \to \tilde{\phi}_\alpha' \to 0 \quad \text{as} \quad y \to \infty \tag{42}$$

if ϕ_α and $\tilde{\phi}_\alpha$ are in M, or the weaker condition

$$\phi_\alpha,\ \phi'_\alpha,\ \tilde{\phi}_\alpha,\ \tilde{\phi}'_\alpha \quad \text{bounded as} \quad y \to \infty, \tag{43}$$

if ϕ_α and $\tilde{\phi}_\alpha$ are in M'. Those eigenfunctions which belong to M will be called discrete eigenfunctions. Those which belong to M' but not M will be called continuum eigenfunctions.

It has been found (Mack 1976; Grosch & Salwen 1975, 1978*a*) that, in general, there is a finite number of discrete eigenfunctions, $\{\phi_{\alpha m}(y)\}$ with eigenvalues $\{\omega_{\alpha m}\}$ and a set, $\{\phi_{\alpha k}\}$, of continuum eigenfunctions with eigenvalues $\{\omega_{\alpha k}\}$ which depend continuously on a real parameter, k, in the range $[0, \infty)$. (Note that the k of this paper is equal to αk of part 1.)

The number of discrete modes, which we shall denote by $N(\alpha)$, depends not only on α but also on R and on the form of $U(y)$ and can, in some cases, be zero. The adjoint eigenfunctions also include a finite set, $\{\tilde{\phi}_{\alpha m}\}$, of discrete eigenfunctions and a continuum, $\{\tilde{\phi}_{\alpha k}\}$, with eigenvalues $\{\omega^*_{\alpha m}\}$ and $\{\omega^*_{\alpha k}\}$, respectively (see discussion following (30)). For a given k, $\phi_{\alpha k}$ and $\tilde{\phi}_{\alpha k}$ vary like a linear combination of $e^{\pm iky}$ as $y \to \infty$. We therefore find that

$$\int_{k-\epsilon}^{k+\epsilon} \phi_{\alpha k'}(y)\, dk' \quad \text{and} \quad \int_{k-\epsilon}^{k+\epsilon} \tilde{\phi}_{\alpha k'}(y)\, dk' \in M \tag{44}$$

and that, for any square-integrable f,

$$(f, \phi_{\alpha k}), \quad (f, \tilde{\phi}_{\alpha k}), \quad \left(f, \frac{d\phi_{\alpha k}}{dy}\right), \quad \left(f, \frac{d\tilde{\phi}_{\alpha k}}{dy}\right) \quad \text{all exist.} \tag{45}$$

Inner products between continuum functions, such as $\langle \tilde{\phi}_{\alpha k}, \phi_{\alpha k'} \rangle$ do not exist in the ordinary sense but are definable in terms of the Dirac δ-function (Lighthill 1960, pp. 10–21).

The discrete eigenvalues must be searched for (Mack 1976), but the continuum eigenvalues follow from the asymptotic form ($\phi_{\alpha k}, \tilde{\phi}_{\alpha k} \sim$ linear combination of $e^{\pm iky}$) of the eigenfunctions as $y \to \infty$ and $U \to U_1 = U(\infty)$,

$$(-k^2-\alpha^2)^2 - (i\alpha R U_1 - iR\omega_{\alpha k})(-k^2-\alpha^2) = 0, \tag{46a}$$

$$(-k^2-\alpha^2)^2 + (i\alpha R U_1 - iR\omega^*_{\alpha k})(-k^2-\alpha^2) = 0, \tag{46b}$$

so that both equations yield

$$\omega_{\alpha k} = -iR^{-1}(k^2+\alpha^2+i\alpha R U_1). \tag{47}$$

We also find that no continuum eigenvalue is also a discrete eigenvalue. Then

$$\langle \tilde{\phi}_{\alpha n}, \phi_{\alpha n'} \rangle = 0 \quad \text{for} \quad \omega_{\alpha n} \neq \omega_{\alpha n'}, \tag{48a}$$

$$\langle \tilde{\phi}_{\alpha n}, \phi_{\alpha k'} \rangle = \langle \tilde{\phi}_{\alpha k}, \phi_{\alpha n'} \rangle = 0, \tag{48b}$$

and

$$\left\langle \tilde{\phi}_{\alpha k}, \int_{k_1}^{k_2} \phi_{\alpha k'}\, dk' \right\rangle = 0 \quad \text{unless} \quad k_1 < k < k_2. \tag{48c}$$

With proper labelling and normalization, it is then possible to choose the eigenfunctions in such a way that

$$\langle \tilde{\phi}_{\alpha n}, \phi_{\alpha n'} \rangle = \delta_{nn'}, \tag{49a}$$

$$\langle \tilde{\phi}_{\alpha n}, \phi_{\alpha k'} \rangle = \langle \tilde{\phi}_{\alpha k}, \phi_{\alpha n'} \rangle = 0, \tag{49b}$$

and

$$\langle \tilde{\phi}_{\alpha k}, \phi_{\alpha k'} \rangle = \delta(k-k'). \tag{49c}$$

3.2. *Expansion of an arbitrary disturbance*

If the eigenfunctions form a complete set, then, for any time t, we may expand $\psi_\alpha(y,t)$ as a linear combination,

$$\psi_\alpha(y,t) = \sum_{n=1}^{N(\alpha)} a_n(\alpha,t)\,\phi_{\alpha n}(y) + \int_0^\infty a_k(\alpha,t)\,\phi_{\alpha k}(y)\,dk \tag{50}$$

of those eigenfunctions. To find the coefficients $\{a_n\}$ and $\{a_k\}$ we may make use of (49) to take inner products

$$\langle\tilde{\phi}_{\alpha n}, \psi_\alpha(y,t)\rangle = \sum_{n'=1}^{N(\alpha)} a_{n'}(\alpha,t)\,\delta_{nn'} = a_n(\alpha,t), \tag{51a}$$

$$\langle\tilde{\phi}_{\alpha k}, \psi_\alpha(y,t)\rangle = \int_0^\infty a_{k'}(\alpha,t)\,\delta(k-k')\,dk' = a_k(\alpha,t). \tag{51b}$$

We then find that

$$\begin{aligned}\frac{\partial a_n(\alpha,t)}{\partial t} &= \left\langle\tilde{\phi}_{\alpha n}, \frac{\partial\psi_\alpha}{\partial t}\right\rangle \\ &= -i\omega_{\alpha n}\langle\tilde{\phi}_{\alpha n}, \psi_\alpha(y,t)\rangle = -i\omega_{\alpha n}a_n(\alpha,t).\end{aligned} \tag{52a}$$

And, similarly,
$$\frac{\partial a_k(\alpha,t)}{\partial t} = \left\langle\tilde{\phi}_{\alpha k}, \frac{\partial\psi_\alpha}{\partial t}\right\rangle = -i\omega_{\alpha k}a_k(\alpha,t), \tag{52b}$$

so that
$$a_n(\alpha,t) = A_n(\alpha)\exp(-i\omega_{\alpha n}t), \quad a_k(\alpha,t) = A_k(\alpha)\exp(-i\omega_{\alpha k}t), \tag{53a, b}$$

where

$$A_n(\alpha) \equiv a_n(\alpha,0) = \langle\tilde{\phi}_{\alpha n}, \psi_\alpha(y,0)\rangle, \quad A_k(\alpha) \equiv a_k(\alpha,0) = \langle\tilde{\phi}_{\alpha k}, \psi_\alpha(y,0)\rangle. \tag{54a, b}$$

Then, referring to equations (32) and (50), we find

$$\begin{aligned}\Psi(x,y,t) = \int_{-\infty}^{\infty}\Bigg\{&\sum_{n=1}^{N(\alpha)} A_n(\alpha)\,\phi_{\alpha n}(y)\exp(-i\omega_{\alpha n}t) \\ &+\int_0^\infty A_k(\alpha)\,\phi_{\alpha k}(y)\exp(-i\omega_{\alpha k}t)\,dk\Bigg\}e^{i\alpha x}\,d\alpha,\end{aligned} \tag{55}$$

where
$$\begin{aligned}A_n(\alpha) &= \langle\tilde{\phi}_{\alpha n}, \psi_\alpha(y,0)\rangle \\ &= -\frac{1}{2\pi}\int_0^\infty \tilde{\phi}^*_{\alpha n}(y)\;L_\alpha\int_{-\infty}^{\infty}\Psi(x,y,0)\,e^{-i\alpha x}\,dx\,dy \\ &= -\frac{1}{2\pi}\int_0^\infty \tilde{\phi}^*_{\alpha n}(y)\int_{-\infty}^{\infty}[\nabla^2\Psi]_{t=0}\,e^{-i\alpha x}\,dx\,dy,\end{aligned} \tag{56a}$$

and, similarly,
$$A_k(\alpha) = -\frac{1}{2\pi}\int_0^\infty \tilde{\phi}^*_{\alpha k}(y)\int_{-\infty}^{\infty}[\nabla^2\Psi]_{t=0}\,e^{-i\alpha x}\,dx\,dy. \tag{56b}$$

If the discrete and continuum eigenfunctions form a complete set, then equation (55) constitutes an expansion of the stream function of an arbitrary disturbance in terms of the discrete (Tollmien–Schlichting) and continuum wave solutions,

$$\phi_{\alpha n}(y)\exp[i(\alpha x - \omega_{\alpha n}t)] \quad\text{and}\quad \phi_{\alpha k}(y)\exp[i(\alpha x - \omega_{\alpha k}t)],$$

of the disturbance equation, (3), with coefficients determined by the initial form of the disturbance $\Psi(x,y,0)$. In the next section we will show that the discrete and continuum eigenfunctions are a complete set.

One interesting and significant result of this calculation is that the initial distribution of vorticity,

$$\zeta_0(x,y) \equiv \zeta(x,y,0) \equiv [\partial v/\partial x - \partial u/\partial y]_{x,y,0} = -\nabla^2\Psi(x,y,0), \tag{57}$$

is sufficient information to determine the coefficients $A_n(\alpha)$ and $A_k(\alpha)$ and, therefore, the subsequent development of the disturbance.

4. Completeness of the temporal expansion functions

Gustavsson (1979) has carried out a formal solution of the initial-value problem of temporal stability for three-dimensional disturbances. He uses the same co-ordinate system as we do with the addition of the z co-ordinate in the cross-stream direction. The formal solution is obtained by taking Fourier transforms in both x and z and a Laplace transform in t, formally solving the Orr–Sommerfeld equation in the transform space, and formally inverting the transforms. If we eliminate the z variation of Gustavsson's solution and his Fourier transform in z (replacing his k by $|\alpha|$) the two solutions should be identical. Both Gustavsson and we express the solution in physical space as an inverse Fourier transform over α, the transform variable in the x direction. In order to show that these two methods yield identical results it is therefore necessary to show that his formal solution in Fourier space, $\hat{v}$, as given in (G13),† is equal to the factor in curly brackets in our equation (55).

In order to do this we must first translate Gustavsson's notation into our notation. Setting $\beta = 0$, after (G3) it is easily seen that we have the following correspondence:

This paper	Gustavsson
Ψ	iv/α
$\lvert\alpha\rvert$	k
ω	is
k	σ
U_1	1

in (G3) and thereafter.

Gustavsson gives the formal solution in Fourier space in equation (G18). It consists of a sum of the residue values at the poles plus a contour integral along a branch cut. Using the definitions of W as the Wronskian, the D_j, given after (G6), and the ϕ_j, equation (G7), it is quite straightforward to show that the residue, R_ν, at a pole s_ν is

$$R_\nu \equiv (e^{s_\nu t}/W) \lim_{s\to s_\nu} \{(s-s_\nu)\,[a_1(s)\,\phi_1(y,s) + a_2(s)\,\phi_2(y,s)]\}. \tag{58}$$

Therefore the residue consists of a linear combination of ϕ_1 and ϕ_2, the solutions of the Orr–Sommerfeld equation that approach zero as $y \to \infty$, i.e. they satisfy (G4) and (42). At $s = s_\nu$, ϕ_1 and ϕ_2 satisfy the usual eigenvalue condition at $y = 0$ for the discrete modes of the Orr–Sommerfeld equation, condition (Gb) (at the bottom of page 1603). This linear combination thus satisfies (41). Therefore the residue at s_ν is proportional to our discrete eigenfunction $\phi_{\alpha\nu}(y)$ with eigenvalue $\omega_{\alpha\nu}$, and

$$e^{s_\nu t} \equiv \exp(-i\omega_{\alpha\nu} t).$$

It is well known (Coddington & Levinson 1955, p. 101, problem 19) that $[D_j/W]^*$, the complex conjugates of the functions used in (G6), are solutions of the adjoint

† In order to simplify reference to the equations in Gustavsson's paper we will hereafter use the prefix G. Page references are also to Gustavsson's paper.

equation (39). It can be seen from the form of (G11) and the definition of our inner product (23) that

$$a_j = \langle D_j^*, \phi_0 \rangle, \quad j = 3, 4, \tag{59}$$

so that the coefficient of $\phi_{\alpha\nu}(y)$ in the residue is the inner product of *some* solution of the adjoint equation with $\phi(y, 0)$. Finally, some straightforward, but tedious, algebra shows that the particular linear combination of the D_j^* involved satisfies the boundary conditions (41) and (42) and therefore is a multiple of our $\tilde{\phi}_{\alpha\nu}$. We thus find that the residue at s_ν is

$$R_\nu = d_{\alpha\nu} A_\nu(\alpha)\, \phi_{\alpha\nu} \exp(-i\omega_{\alpha\nu} t), \tag{60}$$

with $d_{\alpha\nu}$ independent of y and $A_\nu(\alpha)$ given by (54*a*). Before determining $d_{\alpha\nu}$, we turn to the contribution of the branch cut.

Using the fact that our $\omega = is$, it is clear from (G14) that the branch cut in the complex s plane is our continuous spectrum in the complex ω (or c) plane and that the branch point, $\mu = 0$, corresponds to the limit point of our continuous spectrum at $c = U_1 - i\alpha^2/R$, with $U_1 = 1$. The function $F(\alpha, k; y)$ in (G18) is, by (G17) and (G19), a linear combination of the solutions of the Orr–Sommerfeld equation which are, as $y \to \infty$, asymptotic to $e^{-\alpha y}$, e^{-iky}, and e^{+iky}. It can be shown, using (G19), (G20), (G21), and (G22), that

$$F(\alpha, k; 0) = (dF/dy)_{y=0} = 0, \tag{61}$$

and so $F(\alpha, k; y)$ is some multiple of our continuum eigenfunction $\phi_{\alpha k}(y)$. Further, it is obvious that, in (55),

$$\exp(-i\omega_{\alpha k} t) = \exp(-i\alpha U_1 t) \exp(-(\alpha^2 + k^2)\, t/R) \tag{62}$$

in (G18) with $U_1 = 1$.

Just as for the discrete modes, the $\{a_\nu^I\}$, $\nu = 2, 3, 4$, in (G21) are the inner product of some solutions of the adjoint equation with ϕ_0. Using the definition of the E_{mn} in (G22) and the definitions of the Ψ_m as given in the next to last paragraph on page 1604, some algebra shows that the particular linear combination satisfies the boundary conditions at $y = 0$ and so the inner product in (G20) is a multiple of the inner product of our continuum adjoint, $\tilde{\phi}_{\alpha k}(y)$, with the initial condition. Therefore, the integral term in (G18) is

$$I = \int_0^\infty d_{\alpha k} A_k(\alpha)\, \phi_{\alpha k}(y) \exp(-i\omega_{\alpha k} t)\, dk, \tag{63}$$

with $A_k(\alpha)$ given by (54*b*) and $d_{\alpha k}$ independent of y. Gustavsson's result (G18) thus takes the form (in our notation)

$$\psi_\alpha(y, t) = \sum_{\nu=1}^{N(\alpha)} d_{\alpha\nu} A_\nu(\alpha)\, \phi_{\alpha\nu}(y) \exp(-i\omega_{\alpha\nu} t) + \int_0^\infty d_{\alpha k} A_k(\alpha)\, \phi_{\alpha k}(y) \exp(-i\omega_{\alpha k} t)\, dk. \tag{64}$$

Both Gustavsson and we may choose our initial condition arbitrarily, provided that the various integrals of this function with the adjoint functions exist. If we choose the initial condition that $\psi_\alpha(y, 0)$ is one of the discrete eigenfunctions, say $\phi_{\alpha m}(y)$, then in (55)

$$A_n(\alpha) = \delta_{nm}, \quad A_k(\alpha) = 0. \tag{65a, b}$$

In Gustavsson's formulation (63) we have

$$d_{\alpha\nu} A_\nu(\alpha) = \delta_{\nu m}, \quad A_k(\alpha) \equiv 0. \tag{66a, b}$$

We thus see that

$$d_{\alpha\nu} \equiv 1. \tag{67}$$

If we then choose the initial condition

$$\psi_\alpha(y,0) = \int_{k-\epsilon}^{k+\epsilon} \phi_{\alpha k'}(y)\, dk', \tag{68}$$

a similar argument shows that

$$d_{\alpha k} \equiv 1. \tag{69}$$

Substitution of $d_{\alpha\nu} = d_{\alpha k} = 1$, (67) and (69), makes (64), derived from Gustavsson's solution, identical with the curly bracket in our expansion solution (55). We have thus shown that the formal solution obtained by Gustavsson from the Fourier–Laplace transform is identical, term by term, to our formal expansion solution.

Since any square-integrable solution possesses a Fourier–Laplace expansion, we have shown that our expansion (55) is complete whenever it is valid to separate the Fourier–Laplace transform solution into a sum over the poles plus an integral over the branch cut – that is, whenever the sum over the poles (discrete eigenvalues) converges. This is, of course, also the condition for the validity of Gustavsson's solution.

For the Blasius boundary layer, the numerical evidence (Mack 1976) indicates that, at a given R and α, the number of discrete modes is *finite*, so that the sum over the poles is a finite sum. If this is so, then the above condition is certainly satisfied and our expansion functions form a complete set.

We have shown that the Fourier–Laplace transform result and the eigenfunction expansion result are different forms of the *same* solution of the initial-value problem to be chosen according to convenience in a particular case. The eigenfunction expansion formulation gives explicit formulae (54*a, b*) to calculate the expansion coefficients. This allows one to calculate the amplitudes of the discrete modes (TS modes) and the continuum functions, given the initial distribution of vorticity.

5. The spatial stability problem

5.1. *The eigenvalues and eigenfunctions*

The finiteness condition, equation (7), is modified for the spatial stability problem to

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left\{\left|\frac{\partial\Psi}{\partial x}\right|^2 + \left|\frac{\partial\Psi}{\partial y}\right|^2\right\} dt\, dy < \infty. \tag{7''}$$

This ensures that the Fourier integral expansion of Ψ,

$$\Psi(x,y,t) = \int_{-\infty}^{\infty} \psi_\omega(x,y)\, e^{-i\omega t}\, d\omega \tag{70}$$

exists. If we assume that ψ_ω is of the form

$$\psi_\omega(x,y) = \phi_\omega(y)\, e^{i\alpha x}, \tag{71}$$

then ϕ_ω is the solution of the Orr–Sommerfeld equation

$$\left\{L_\alpha^2 - iR\left[(\alpha U - \omega)\, L_\alpha - \alpha \frac{d^2U}{dy^2}\right]\right\}\phi_\omega = 0, \tag{72}$$

with L_α given by (36).

Similarly, we assume that the adjoint solution, $\tilde{\Psi}$, also satisfies equation (7″), thus ensuring that

$$\tilde{\Psi}(x,y,t) = \int_{-\infty}^{\infty} \tilde{\psi}_\omega(x,y)\, e^{-i\omega t}\, d\omega \tag{73}$$

exists. We assume that

$$\tilde{\psi}_\omega(x,y) = \tilde{\phi}_\omega(y)\, e^{i\alpha^* x}. \tag{74}$$

Then $\tilde{\phi}_\omega$ is the solution of the adjoint Orr–Sommerfeld equation

$$\left\{L^2_{\alpha^*} + iR\left[(\alpha^* U - \omega)\, L_{\alpha^*} + 2\alpha^* \frac{dU}{dy}\frac{d}{dy}\right]\right\}\tilde{\phi}_\omega = 0. \tag{75}$$

The boundary conditions are

$$\phi_\omega(0) = \phi'_\omega(0) = \tilde{\phi}_\omega(0) = \tilde{\phi}'_\omega(0) = 0, \tag{76}$$

and

$$\phi_\omega \to \phi'_\omega \to \tilde{\phi}_\omega \to \tilde{\phi}'_\omega \to 0 \quad \text{as} \quad y \to \infty, \tag{77}$$

if ϕ_ω and $\tilde{\phi}_\omega$ are in M, or

$$\phi_\omega,\ \ \phi'_\omega,\ \ \tilde{\phi}_\omega,\ \ \tilde{\phi}'_\omega \ \ \text{bounded as} \quad y \to \infty, \tag{78}$$

if ϕ_ω and $\tilde{\phi}_\omega$ are in M'. As above, the eigenfunctions which belong to M are the discrete eigenfunctions and those that belong to M' but not M are the continuum eigenfunctions.

Jordinson (1971), Corner, Houston & Ross (1976), and Murdock & Stewartson (1977) have shown that there is only a finite set of discrete eigenfunctions, $\{\phi_{\omega n}(y)\}$, with eigenvalues $\{\alpha_{\omega n}\}$. The set of discrete adjoint eigenfunctions, $\{\tilde{\phi}_{\omega n}\}$, with eigenvalues $\{\alpha^*_{\omega n}\}$ is also finite. The number of discrete modes, $N(\omega)$, depends on R as well as ω and can be zero.

In part 1 we showed that, in an unbounded domain, the spatial stability problem always has a continuous spectrum. Since then we have discovered (Grosch & Salwen 1978*b*), that the spatial continuum of part 1 is only one branch of a four-branched spatial continuum. It is quite easy to show the existence of the four branches of the spatial continuum. We look for solutions to equations (72) and (75), $\phi_{\omega k}(y)$ and $\tilde{\phi}_{\omega k}(y)$, for a given real k, which vary like $e^{\pm iky}$ as $y \to \infty$ (the k used in discussing the spatial continuum in part 1 is $2/R$ times the k used here). Noting that, as $y \to \infty$, $U \to U_1$, a constant, and U', $U'' \to 0$, we have

$$(-\alpha^2 - k^2)(-\alpha^2 - k^2 - i\alpha R U_1 + i\omega R) = 0, \tag{79a}$$

and

$$(-\alpha^{*2} - k^2)(-\alpha^{*2} - k^2 + i\alpha^* R U_1 - i\omega R) = 0. \tag{79b}$$

(Note that equations (79*a* and *b*) are complex conjugates.)

It is obvious that there are four roots, $\{\alpha_j\}, j = 1, \ldots, 4$, with α_1 and α_2 the roots of

$$\alpha_j^2 + iRU_1\alpha_j + k^2 - i\omega R = 0 \tag{80}$$

and

$$\alpha_3 = ik, \quad \alpha_4 = -ik. \tag{81a, b}$$

The eigenvalue α_1, the root of equation (80) with positive real part, is the continuum eigenvalue discussed in part 1. As was discussed in part 1, the eigenfunctions of this branch of the spatial continuum are waves propagating in the downstream $(+x)$ direction and decaying in amplitude as they travel. In the same way it can be shown that α_2, is the eigenvalue of a continuum eigenfunction which is a wave travelling in the upstream $(-x)$ direction and decaying as it travels.

The free-stream speed, U_1, can be taken to be unity for a boundary layer, wake, or free shear flow. In most cases of interest $\omega/R \ll 1$. It is easy to see that, with $U_1 = 1$, and $\omega/R \ll 1$,

$$\alpha_{1,2} = \pm(\omega/\gamma) - iR[\tfrac{1}{2}(1 \mp \gamma) \mp (\omega/R)^2/\gamma^3] + O(\omega^3/R^2) \tag{82}$$

with

$$\gamma = (1 + 4k^2/R^2)^{\frac{1}{2}}. \tag{83}$$

Define, as usual, the phase speed c_j by

$$c_j = \alpha_j^* \omega/|\alpha_j|^2. \tag{84}$$

Then as $k \to 0$,

$$\alpha_1 \approx \omega + iR[(\omega^2 + k^2)/R^2], \tag{85a}$$

$$\alpha_2 \approx -\omega - iR[1 + (\omega^2 + k^2)/R^2], \tag{85b}$$

$$c_1 \approx 1 - i[(\omega^2 + k^2)/R^2]/(\omega/R), \tag{86a}$$

$$c_2 \approx -(\omega/R)^2 + i(\omega/R)[1 + (\omega^2 + k^2)/R^2]; \tag{86b}$$

while, as $k \to \infty$,

$$\alpha_1 \approx (\omega R/2k) + ik, \quad \alpha_2 \approx -\alpha_1, \tag{87a, b}$$

$$c_1 \approx \omega^2 R/2k^3 - i\omega/k, \quad c_2 \approx -c_1. \tag{88a, b}$$

The damping rate, for the spatial eigenfunctions, is $\mathrm{Im}(\alpha)$ and the phase speed is $\mathrm{Re}(c)$. The equations given above show that the eigenfunctions on branch 2 of the spatial continuum, for boundary layers, wakes, and free shear flows, always have both a very large damping rate and a very small phase speed. This is in marked contrast to those of branch 1, which, as was shown in part 1, or can be seen from the above results, contains lightly damped eigenfunctions some of which have a very slow phase speed and some of which have a phase speed nearly equal to the free-stream speed.

The spatial continuum eigenfunctions of branches 3 and 4 are standing waves in x because they vary like

$$\exp(i\alpha_3 x) = \exp(-kx), \quad \exp(i\alpha_4 x) = \exp(+kx). \tag{89a, b}$$

As in the temporal case, the inner products between the spatial continuum eigenfunctions do not exist in the ordinary sense but can be defined as δ functions. Then, with proper labelling and normalization, it is possible to choose the eigenfunctions such that (with the superscript i or j indicating the branch of the continuum)

$$[\![\tilde{\phi}_{\omega n}, \phi_{\omega n'}]\!] = \delta_{nn'}, \tag{90a}$$

$$[\![\tilde{\phi}_{\omega n}, \phi_{\omega k}^{(i)}]\!] = [\![\tilde{\phi}_{\omega k}^{(i)}, \phi_{\omega n}]\!] = 0, \tag{90b}$$

$$[\![\tilde{\phi}_{\omega k}^{(i)}, \phi_{\omega k'}^{(j)}]\!] = \delta(k - k')\,\delta_{ij}, \tag{90c}$$

where

$$[\![\tilde{\phi}_{\omega n}, \phi_{\omega n'}]\!] \equiv \int_0^\infty \left\{\left[\frac{-i(\alpha_{\omega n'} + \alpha_{\omega n})}{R}\right]\left[(\alpha_{\omega n'}^2 + \alpha_{\omega n}^2 - i\omega R)\,\tilde{\phi}_{\omega n}^*\phi_{\omega n'} + 2\frac{d\tilde{\phi}_{\omega n}^*}{dy}\frac{d\phi_{\omega n'}}{dy}\right]\right.$$
$$\left. + U\left[(\alpha_{\omega n'}^2 + \alpha_{\omega n'}\alpha_{\omega n} + \alpha_{\omega n}^2)\,\tilde{\phi}_{\omega n}^*\phi_{\omega n'} + 2\frac{d\tilde{\phi}_{\omega n}^*}{dy}\frac{d\phi_{\omega n'}}{dy} + \frac{d^2\tilde{\phi}_{\omega n}^*}{dy^2}\phi_{\omega n'}\right]\right\}dy, \tag{91}$$

and there are analogous expressions for the inner products in (90b) and (90c).

5.2. *Expansion of an arbitrary disturbance*

If the spatial eigenfunctions form a complete set, then, for any x, we may expand $\psi_\omega(x,y)$ as

$$\psi_\omega(x,y) = \sum_{n=1}^{N(\omega)} a_n(\omega,x)\,\phi_{\omega n}(y) + \sum_{i=1}^{4}\int_0^\infty a_k^{(i)}(\omega,x)\,\phi_{\omega k}^{(i)}(y)\,dk. \tag{92}$$

In order to find the coefficients $\{a_n(\omega,x)\}$ and $\{a_k^{(i)}(\omega,x)\}$ we use equation (90) to take the inner products

$$[\![\tilde{\phi}_{\omega n}, \psi_\omega(x,y)]\!] = \sum_{n'=1}^{N(\omega)} a_{n'}(\omega,x)\,\delta_{nn'} = a_n(\omega,x), \tag{93a}$$

$$[\![\tilde{\phi}_{\omega k}^{(i)}, \psi_\omega(x,y)]\!] = \int_0^\infty a_{k'}^{(j)}(\omega,x)\,\delta(k-k')\,\delta_{ij}\,dk' = a_k^{(i)}(\omega,x). \tag{93b}$$

Then

$$\frac{\partial a_n(\omega,x)}{\partial x} = \left[\!\!\left[\tilde{\phi}_{\omega n}, \frac{\partial\psi_\omega}{\partial x}\right]\!\!\right] = i\alpha_{\omega n}[\![\tilde{\phi}_{\omega n}, \psi_\omega]\!]$$

$$= i\alpha_{\omega n}\,a_n(\omega,x), \tag{94a}$$

and

$$\frac{\partial a_k^{(i)}(\omega,x)}{\partial x} = \left[\!\!\left[\tilde{\phi}_{\omega k}, \frac{\partial\psi_\omega}{\partial x}\right]\!\!\right] = i\alpha_{\omega k}^{(i)}[\![\tilde{\phi}_{\omega k}, \psi_\omega]\!] = i\alpha_{\omega k}^{(i)}a_k^{(i)}(\omega,x), \tag{94b}$$

so that

$$a_n(\omega,x) = A_n(\omega)\exp(i\alpha_{\omega n}x), \tag{95a}$$

$$a_k^{(i)}(\omega,x) = A_k^{(i)}(\omega)\exp(i\alpha_{\omega k}^{(i)}x), \tag{95b}$$

where

$$A_n(\omega) \equiv a_n(\omega,0) = [\![\tilde{\phi}_{\omega n}, \psi_\omega(0,y)]\!], \tag{96a}$$

$$A_k^{(i)}(\omega) \equiv a_k^{(i)}(\omega,0) = [\![\tilde{\phi}_{\omega k}^{(i)}, \psi_\omega(0,y)]\!]. \tag{96b}$$

From equations (73), (92) and (95), we have the formal solution to the spatial stability problem for the two-dimensional, linearized Navier–Stokes equations

$$\Psi(x,y,t) = \int_{-\infty}^{\infty}\left\{\sum_{n=1}^{N(\omega)} A_n(\omega)\,\phi_{\omega n}(y)\exp(i\alpha_{\omega n}x) + \sum_{i=1}^{4}\int_0^\infty A_k^{(i)}(\omega)\,\phi_{\omega k}^{(i)}(y)\exp(i\alpha_{\omega k}^{(i)}x)\,dk\right\}\exp(-i\omega t)\,d\omega. \tag{97}$$

Define

$$\psi_\omega^{(0)}(y) \equiv \psi_\omega(0,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(0,y,t)\,e^{i\omega t}\,dt, \tag{98a}$$

$$\psi_\omega^{(1)}(y) \equiv \left(\frac{\partial\psi_\omega(x,y)}{\partial x}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial\Psi(x,y,t)}{\partial x}\right)_{x=0} e^{i\omega t}\,dt, \tag{98b}$$

$$\psi_\omega^{(2)}(y) \equiv \left(\frac{\partial^2\psi_\omega(x,y)}{\partial x^2}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi(x,y,t)}{\partial x^2}\right)_{x=0} e^{i\omega t}\,dt, \tag{98c}$$

$$\psi_\omega^{(3)}(y) \equiv \left(\frac{\partial^3\psi_\omega(x,y)}{\partial x^3}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial^3\Psi(x,y,t)}{\partial x^3}\right)_{x=0} e^{i\omega t}\,dt. \tag{98d}$$

Then

$$A_n(\omega) = [\![\tilde{\phi}_{\omega n}, \psi_\omega(0,y)]\!]$$

$$= \int_0^\infty \Bigg\{ \frac{1}{R}\Bigg[\tilde{\phi}^*_{\omega n}\psi^{(3)}_\omega + i\alpha_{\omega n}\tilde{\phi}^*_{\omega n}\psi^{(2)}_\omega - \alpha^2_{\omega n}\tilde{\phi}^*_{\omega n}\psi^{(1)}_\omega - i\alpha^3_{\omega n}\tilde{\phi}^*_{\omega n}\psi^{(0)}_\omega - 2i\alpha_{\omega n}\frac{d\tilde{\phi}^*_{\omega n}}{dy}\frac{\partial\psi^{(0)}_\omega}{\partial y}$$

$$-2\frac{d\tilde{\phi}^*_{\omega n}}{dy}\frac{d\psi^{(1)}_\omega}{dy}\Bigg] - i\omega[\tilde{\phi}^*_{\omega n}\psi^{(1)}_\omega - i\alpha\tilde{\phi}^*_\omega\psi^{(0)}_{\omega n}]$$

$$-U\Bigg[\tilde{\phi}^*_{\omega n}\psi^{(2)}_\omega + i\alpha_{\omega n}\tilde{\phi}^*_{\omega n}\psi^{(1)}_\omega - \alpha^2_{\omega n}\tilde{\phi}^*_{\omega n}\psi^{(0)}_\omega - 2\frac{d\tilde{\phi}^*_{\omega n}}{dy}\frac{d\psi^{(0)}_\omega}{dy} - \frac{d^2\tilde{\phi}^*_{\omega n}}{dy^2}\psi^{(0)}_\omega\Bigg]\Bigg\}\,dy, \quad (99)$$

and there is a similar expression for $A^{(i)}_k(\omega)$.

This is the formal solution of the spatial stability problem for an arbitrarily imposed boundary condition at $x = 0$. The boundary conditions which must be specified are the Fourier transforms, in time, of the stream function and its first three partial derivatives with respect to x, evaluated at $x = 0$.

As it stands, this formal solution will not give a physically acceptable solution because, given an arbitrary $\Psi(0,y,t)$ and derivatives, disturbances which lie on all four branches of the continuum will be excited. Therefore the solution will contain, in addition to the waves propagating towards $x = \infty$ and the standing waves whose amplitude decays towards $x = \infty$, waves propagating *upstream* from $x = \infty$ and standing waves whose amplitude *increases* towards $x = \infty$.

A condition must be imposed that, for $x > 0$, all propagating disturbances are travelling in the positive x direction and all standing waves have amplitudes which decay in the positive x direction. It appears that this should be done by requiring that $\Psi(0,y,t)$ and its first three partial derivatives with respect to x be orthogonal to all eigenfunctions on branches 2 and 4 of the continuous spectrum but we have not yet investigated the implications of imposing this condition on the disturbance stream function at $x = 0$.

6. Application to the temporal development of a model flow

In this section, we apply the results of § 3 to the simple base flow,

$$U(y) = U_1 = \text{constant}, \quad y \geqslant 0, \quad (100)$$

which is a slip flow past a bounding plane at $y = 0$. Though the base flow velocity does not vanish at the boundary, we still require the disturbance velocity to be zero at $y = 0$. Because of the simplicity of the base flow, the expansion functions are elementary functions. In § 6.1, we find the expansion functions. In this case, there are no discrete eigenmodes; all of the eigenfunctions are continuum functions.

In § 6.2, we solve the the time development of a particular initial disturbance by expanding in terms of these eigenfunctions. The initial disturbance chosen is a periodic layer of vorticity confined to a plane parallel to the ($y = 0$) boundary. Because of the simple form of the initial disturbance and the simplicity of the base flow, it is possible to obtain the solution in closed form in terms of error functions.

6.1. *The eigenfunctions*

For the base flow of equation (100) the differential equation, (34), for the expansion functions becomes

$$\left(\frac{d^2}{dy^2}-\alpha^2-i\alpha R(U_1-c)\right)\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi = 0, \tag{101}$$

with the general solution (for $\alpha \neq 0, c \neq U_1$),

$$\phi = A\,e^{-|\alpha|y}+B\,e^{|\alpha|y}+C\,e^{py}+D\,e^{-py}, \tag{102}$$

where

$$p^2 = \alpha^2+i\alpha R(U_1-c). \tag{103}$$

(In this case of constant U, $\tilde{\phi}$ must satisfy the *same* differential equation.) In addition, ϕ must satisfy (11) and (13); i.e. ϕ and ϕ' must vanish at $y = 0$ and be bounded in $[0,\infty)$. Since $e^{|\alpha|y}$ is unbounded, $B = 0$. To satisfy the boundary condition at the origin, we must then have

$$\phi = A[e^{-|\alpha|y}-\cosh py+|\alpha|\,p^{-1}\sinh py], \tag{104}$$

which is unbounded as $y \to \infty$ unless p is purely imaginary. The solutions are then given by

$$p = ik; \quad 0 < k < \infty, \tag{105}$$

$$\omega_{\alpha k} = -iR^{-1}(\alpha^2+k^2+i\alpha RU_1), \tag{47}$$

$$\phi_{\alpha k}(y) = \tilde{\phi}_{\alpha k}(y) = A_{\alpha k}[(e^{-|\alpha|y}-\cos ky)+|\alpha|\,k^{-1}\sin ky], \tag{106}$$

where the normalization constant,

$$A_{\alpha k} = k(k^2+\alpha^2)^{-1}\,(2/\pi)^{\frac{1}{2}}, \tag{107}$$

is determined by the condition

$$\langle \tilde{\phi}_{\alpha k}, \phi_{\alpha k'}\rangle = \delta(k-k'). \tag{108}$$

In this case, where the $\phi_{\alpha k}$ and $\tilde{\phi}_{\alpha k}$ are known explicitly, one may show directly that, for $F(y)$ any continuous, differentiable, square-integrable function in $[0,\infty)$,

$$\int_0^\infty \langle \tilde{\phi}_{\alpha k}, F\rangle\, \phi_{\alpha k}(y)\,dk = F(y)-\exp\left(-|\alpha|y\right)F(0), \tag{109}$$

thus confirming that the set of $\{\phi_{\alpha k}\}$ is complete for functions in M, with $F(0) = 0$.

6.2. *The temporal evolution of an initial disturbance*

In order to demonstrate the application of this expansion technique, we consider the particular initial disturbance

$$-\nabla^2\Psi(x,y,0) = \zeta(x,y,0) = \zeta_0\exp\left(i\alpha_0 x\right)\delta(y-y_0), \tag{110}$$

a periodic layer of vorticity at a distance y_0 from the boundary. Following § 3.2, we find that the stream function at any time will be given by

$$\Psi(x,y,t) = \int_{-\infty}^{\infty}\int_0^\infty A_k(\alpha)\,\phi_{\alpha k}(y)\exp\left(-i\omega_{\alpha k}t\right)dk\exp\left(i\alpha x\right)d\alpha, \tag{111}$$

where

$$A_k(\alpha) = \frac{1}{2\pi}\int_0^\infty \tilde{\phi}_{\alpha k}(y)\int_{-\infty}^{\infty}\zeta(x,y,0)\exp\left(-i\alpha x\right)dx\,dy. \tag{112}$$

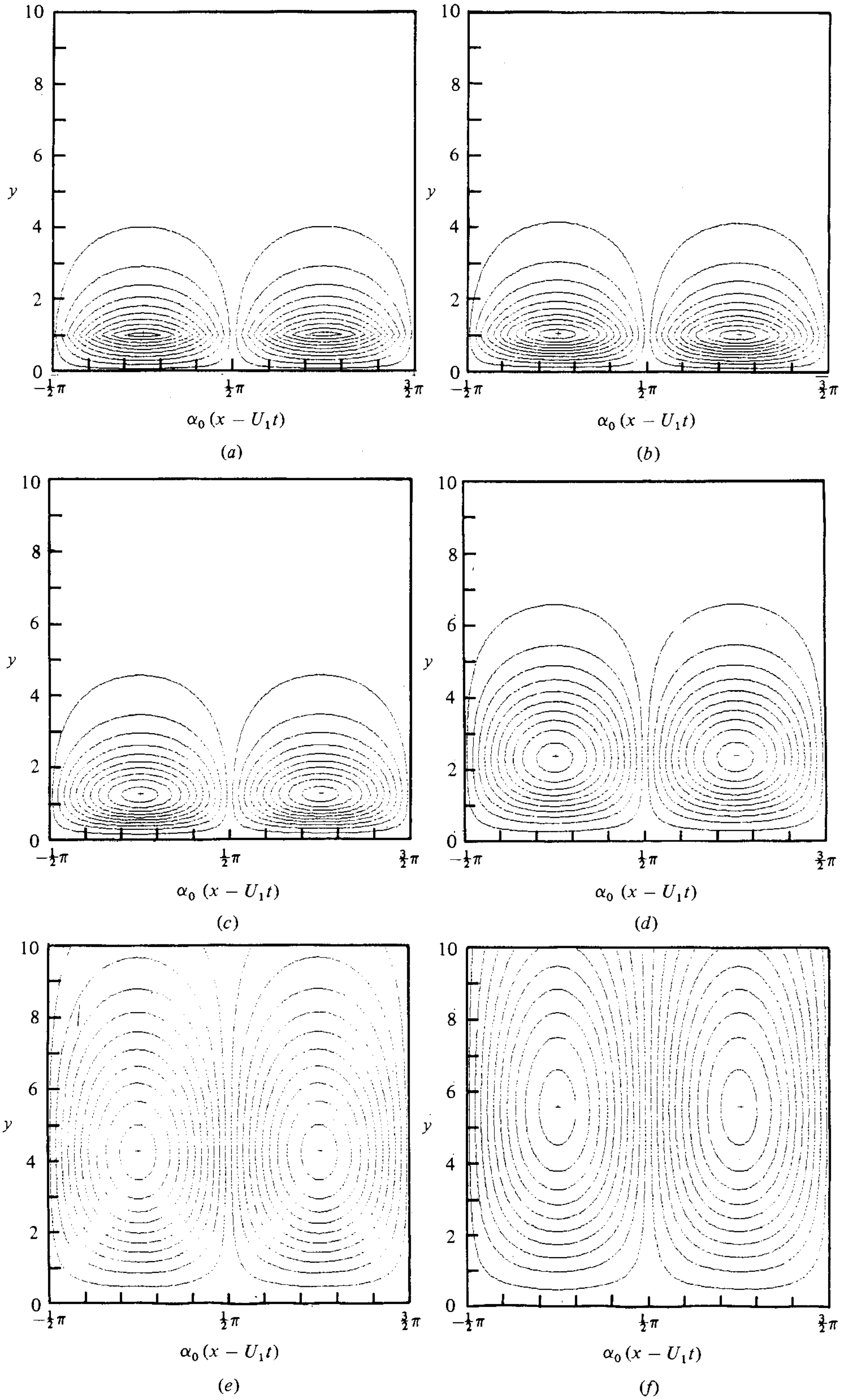

FIGURE 1. For legend see opposite.

It is easily seen, by substituting (110) into (112), that

$$A_k(\alpha) = \phi_{\alpha k}(y_0)\,\delta(\alpha-\alpha_0), \tag{113}$$

so that

$$\begin{aligned}\Psi(x,y,t) &= \int_{-\infty}^{\infty}\int_0^{\infty} \phi_{\alpha k}(y_0)\,\delta(\alpha-\alpha_0)\,\phi_{\alpha k}(y)\exp(-i\omega_{\alpha k}t)\,dk\exp(i\alpha x)\,d\alpha \\ &= \exp(i\alpha_0(x-U_1t))\exp(-\alpha_0^2t/R)\int_0^{\infty} \phi_{\alpha_0 k}(y_0)\,\phi_{\alpha_0 k}(y)\exp(-k^2t/R)\,dk. \end{aligned} \tag{114}$$

After using (106) and (107), for $\phi_{\alpha_0 k}$, we find that each term in the integral is expressible as sums of error functions. The results are given in an appendix. From these results, it can be shown that, for $t \to \infty$ with y fixed,

$$\Psi \sim t^{-\frac{1}{2}}\exp(-\alpha_0^2\,t/R)\cos\alpha_0(x-U_1t)\times(\text{function of } y) \tag{115}$$

and, for $y \to \infty$ with t fixed,

$$\Psi \sim \exp(-\alpha_0(y+y_0))\cos\alpha_0(x-U_1t)\times(\text{function of } t). \tag{116}$$

It is clear that, even though the individual eigenfunctions used in the expansion oscillate with constant amplitude as $y \to \infty$, the *wave packet* behaves like $e^{-\alpha_0 y}$ as $y \to \infty$.

Figure 1 shows contour plots of the stream function for the disturbance, in a frame of reference moving with the free-stream velocity, at six different times. We have chosen $\alpha_0 = 1{\cdot}0$ and $y = 1{\cdot}0$ for the example shown here. Contours of the disturbance stream function have also been calculated for other combinations of values of α_0 and y_0 and, for these other values, the evolution of the disturbance in time is quite similar to that shown in figure 1.

In figure 1 the (+) and (−) indicate the position of the maximum and minimum values of the stream function. These maximum and minimum values are given in the caption to the figure. The flow is counter-clockwise around a maximum (+) and clockwise around a minimum (−).

It is clear from this figure that the disturbance, which is a periodic vortex sheet at $t = 0$, retains its identity as a periodic array for all time, but as time increases it diffuses, the strength decays, and the centres of the vortices drift away from the boundary at $y = 0$.

We could, of course, generalize this model problem by considering an initial vorticity distribution in the y direction. We have not carried out this calculation because our intention in solving this model problem was to illustrate the expansion procedure and we do not think that it warrants further elaboration.

FIGURE 1. Contours of the disturbance stream function for the model problem in a frame of reference moving with the free-stream velocity at six different times. In this example $\zeta_0 = 1{\cdot}0$, $\alpha_0 = 1{\cdot}0$, and $y_0 = 1{\cdot}0$. There are twenty contour lines on each plot. The values of Ψ on these contours are $0{\cdot}95\Psi_{max}$, $0{\cdot}85\Psi_{max}$, ..., $-0{\cdot}95\Psi_{max}$. The (+) and (−) indicate the positions where $\Psi = \Psi_{max}$ and Ψ_{min}. Note that $\Psi_{min} = -\Psi_{max}$. (*a*) $t/R = 10^{-3}$, $\Psi_{max} = 0{\cdot}425$. (*b*) $t/R = 10^{-2}$, $\Psi_{max} = 0{\cdot}359$. (*c*) $t/R = 10^{-1}$, $\Psi_{max} = 0{\cdot}212$. (*d*) $t/R = 1{\cdot}0$, $\Psi_{max} = 0{\cdot}228\times10^{-1}$. (*e*) $t/R = 5{\cdot}0$, $\Psi_{max} = 0{\cdot}108\times10^{-3}$. (*f*) $t/R = 10{\cdot}0$, $\Psi_{max} = 0{\cdot}383\times10^{-6}$.

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Appendix. Solution of the model problem

In § 6 we showed that the stream function for the model problem is, equation (114),

$$\Psi(x,y,t) = \exp(i\alpha_0(x-U_1t))\exp(-i\alpha_0^2t/R)\int_0^\infty \exp(-k^2t/R)\,\phi_{\alpha_0k}(y_0)\,\phi_{\alpha_0k}(y)\,dk,$$

where $\phi_{\alpha_0k}(y)$ is given by (106) and (107). Substituting for $\phi_{\alpha_0k}(y)$ and $\phi_{\alpha_0k}(y_0)$ in this integral it is straightforward to show that, with $\tau = t/R$,

$$\begin{aligned}\Psi(x,y,t) = \exp(i\alpha_0(x-U_1t))\{&I_1(\alpha_0,\tau)-\exp(\alpha_0y)\,I_2(\alpha_0,\tau,y_0)\\&+\alpha_0\exp(-\alpha_0y)\,I_3(\alpha_0,\tau,y_0)-\exp(-\alpha_0y_0)\,I_2(\alpha_0,\tau,y)\\&+\tfrac{1}{2}I_2(\alpha_0,\tau,y+y_0)+\tfrac{1}{2}I_2(\alpha_0,\tau,y-y_0)\\&-\alpha_0I_3(\alpha_0,\tau,y+y_0)+\alpha_0\exp(-\alpha_0y_0)\,I_3(\alpha_0,\tau,y)\\&+\tfrac{1}{2}\alpha_0^2I_4(\alpha_0,\tau,y-y_0)-\tfrac{1}{2}\alpha_0^2I_4(\alpha_0,\tau,y+y_0)\},\end{aligned} \tag{A 1}$$

where the functions I_j are given by

$$\begin{aligned}I_1(\alpha,\tau) &\equiv \frac{2}{\pi}\exp(-\alpha^2\tau)\int_0^\infty \exp(-k^2\tau)\left[\frac{k^2}{(k^2+\alpha^2)^2}\right]dk\\&= \alpha^{-1}\left[(\tfrac{1}{2}+\alpha^2\tau)\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}})-\frac{\alpha\tau^{\frac{1}{2}}}{\sqrt{\pi}}\exp(-\alpha^2\tau)\right],\end{aligned} \tag{A 2}$$

$$\begin{aligned}I_2(\alpha,\tau,Z) &\equiv \frac{2}{\pi}\exp(-\alpha^2\tau)\int_0^\infty \exp(-k^2\tau)\left[\frac{k^2}{(k^2+\alpha^2)^2}\right]\cos kZ\,dk\\&= \frac{1}{4\alpha}[(1+2\alpha^2\tau-\alpha Z)\,e^{-\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}-\alpha Z/2\alpha\tau^{\frac{1}{2}})\\&\quad+(1+2\alpha^2\tau+\alpha Z)\,e^{\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}+\alpha Z/2\alpha\tau^{\frac{1}{2}})\\&\quad-4\alpha\tau^{\frac{1}{2}}\pi^{-\frac{1}{2}}\exp(-\alpha^2\tau)\exp(-\alpha^2Z^2/4\alpha^2\tau)],\end{aligned} \tag{A 3}$$

$$\begin{aligned}I_3(\alpha,\tau,Z) &\equiv \frac{2}{\pi}\exp(-\alpha^2\tau)\int_0^\infty \exp(-k^2\tau)\left[\frac{k}{(k^2+\alpha^2)^2}\right]\sin kZ\,dk\\&= \frac{1}{4\alpha^2}[(2\alpha^2\tau+\alpha Z)\,e^{\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}+\alpha Z/2\alpha\tau^{\frac{1}{2}})\\&\quad-(2\alpha^2\tau-\alpha Z)\,e^{-\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}-\alpha Z/2\alpha\tau^{\frac{1}{2}})],\end{aligned} \tag{A 4}$$

$$\begin{aligned}I_4(\alpha,\tau,Z) &\equiv \frac{2}{\pi}\exp(-\alpha^2\tau)\int_0^\infty \exp(-k^2\tau)\left[\frac{1}{(k^2+\alpha^2)^2}\right]\cos kZ\,dk\\&= \frac{1}{4\alpha^3}[(1-2\alpha^2\tau+\alpha Z)\,e^{-\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}-\alpha Z/2\alpha\tau^{\frac{1}{2}})\\&\quad+(1-2\alpha^2\tau-\alpha Z)\,e^{\alpha Z}\,\mathrm{erfc}(\alpha\tau^{\frac{1}{2}}+\alpha Z/2\alpha\tau^{\frac{1}{2}})\\&\quad+4\alpha\tau^{\frac{1}{2}}\pi^{-\frac{1}{2}}\exp(-\alpha^2\tau)\exp(-\alpha^2Z^2/4\alpha^2\tau)],\end{aligned} \tag{A 5}$$

and, as usual,

$$\mathrm{erfc}(Z) \equiv 2\pi^{-\frac{1}{2}}\int_Z^\infty \exp(-\xi^2)\,d\xi. \tag{A 6}$$

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